

AN IMPROVEMENT IN THE RESTRICTED TERNARY GOLDBACH PROBLEM

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ABSTRACT. We investigate the restricted ternary Goldbach problem, where one of the primes is confined to a short interval. Specifically, we prove that for any sufficiently large odd integer X , the equation $X = p_1 + p_2 + p_3$ is solvable in primes with $p_1 \leq X^{\frac{21}{800} + \varepsilon}$. This is achieved by combining the circle method with minorants and the modern introduction of a non-negative model. Such an approach supersedes the traditional vector-sieve inequality and strictly improves upon the previous result due to Y. Cai. Additionally, we state a general theorem, which separates the minorant construction from the circle method machinery.

1. INTRODUCTION

The Goldbach problem is one of the most famous conjectures in mathematics. Proposed more than 280 years ago in a letter from Christian Goldbach to Leonhard Euler, it states that every even integer greater than 2 is the sum of two primes. Many mathematicians have approached the problem, yet it remains unsolved. The weaker version of the problem, namely the ternary Goldbach conjecture, states that any odd number greater than 5 is a sum of three primes. In 1937, Vinogradov [11] proved that this statement holds for a sufficiently large odd number, which was a breakthrough. Finally, in 2013, Harald Helfgott [7] completely proved the weak conjecture. In the meantime, other mathematicians spent time improving Vinogradov's result by studying the ternary problem with one prime restricted, namely that for a sufficiently large odd integer X , the equation

$$X = p_1 + p_2 + p_3, \quad p_1 \leq H$$

is solvable in primes p_1, p_2, p_3 , with $H = X^{c+\varepsilon}$ for some constant $0 < c < 1$ and any $\varepsilon > 0$. In 1959, Pan [10] proved this result with $c = \frac{1}{4}$. Later, in 1995, Zhan [13] refined this by incorporating a sieving process for almost all short intervals, obtaining $c = \frac{7}{120}$. Over the next few years, Wong [12] and Jia [8] modified the sieving process, producing constants $\frac{7}{216}$ and $\frac{7}{240}$. Most recently, Cai [1] improved the constant to $c = \frac{11}{400}$. The main result of this paper is to adapt the circle method strategy of Grimmelt [5] to prove the following:

Theorem 1.1. *For any $\varepsilon > 0$ and a sufficiently large odd integer X , the equation*

$$X = p_1 + p_2 + p_3, \quad p_1 \leq H$$

is solvable in primes p_1, p_2, p_3 , with $H = X^{\frac{21}{800} + \varepsilon}$.

Note that for comparison,

$$\begin{aligned}\frac{7}{120} &= 0.058333\dots; & \frac{7}{216} &= 0.032407\dots; \\ \frac{7}{240} &= 0.029166\dots; & \frac{11}{400} &= 0.0275; \\ \frac{21}{800} &= 0.02625.\end{aligned}$$

Motivation and notation. Since we need $p_1 + p_2 + p_3 = X$, putting a restriction on one of the primes naturally restricts the other two. Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be the three intervals that contain the three primes. We will specify the intervals at the end of this section. By $\Lambda(n)$ we denote the log-weighted characteristic function of the set of prime numbers. For a function f and set S we denote $f_S = f \cdot \mathbf{1}_S$. So we have

$$\Lambda_S(n) = \begin{cases} \log n & \text{if } n \text{ is prime and } n \in S \\ 0 & \text{else.} \end{cases}$$

The main object we need to study is then

$$(1.1) \quad S(X) = \Lambda_{\mathcal{I}} * \Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}}(X) = \sum_{\substack{p_1 + p_2 + p_3 = X \\ p_1 \in \mathcal{I}, p_2 \in \mathcal{J}, p_3 \in \mathcal{K}}} \log p_1 \log p_2 \log p_3.$$

In this notation, we want to show that

$$S(X) > 0$$

for a sufficiently large X .

We shall prove the theorem using the circle method and by employing the modern non-negative model. For a finitely supported arithmetic function $f : \mathbb{N} \rightarrow \mathbb{C}$, we denote its Fourier series by $\hat{f} : \mathbb{T} \rightarrow \mathbb{C}$ by

$$\hat{f}(\alpha) = \sum_n f(n) e(\alpha n),$$

where, as usual $e(z) = e^{2\pi iz}$. By Fourier inversion and properties of the Fourier transform, we can then write

$$S(X) = \int_{\mathbb{T}} \hat{\Lambda}_{\mathcal{I}}(\alpha) \hat{\Lambda}_{\mathcal{J}}(\alpha) \hat{\Lambda}_{\mathcal{K}}(\alpha) e(-\alpha X) d\alpha.$$

In a standard fashion, we want to use the Farey dissection to divide $\mathbb{T}$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$, and then estimate the contributions coming from the major and minor arcs. However, the prime indicator functions are complicated, so we prefer to work with something simpler, which we can understand in physical space.

Minorants and the old approach. Since we are only interested in a lower bound, the idea is to replace the prime indicator functions with minorants that are easier to understand. As we shall see later, two of the three primes, namely p_2 and p_3 , live in short intervals, so we want to replace the two prime indicators $\Lambda_{\mathcal{J}}, \Lambda_{\mathcal{K}}$ with minorants ν, ω .

This main premise holds by the simple result: if a, b, c are arithmetic functions with $a \geq 0, b \geq c$, then

$$a * b \geq a * c.$$

For now, ignoring the first term in the convolution, by the non-negativity of prime indicators,

$$\Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}} \geq \nu * \Lambda_{\mathcal{K}}.$$

However, the general strategy does not ensure that the minorants are non-negative, so we cannot simply replace $\nu * \Lambda_{\mathcal{K}}$ with $\nu * \omega$.

To avoid this problem, the previous approach of Cai [1] utilised Harman's idea of using two minorants ν_1, ω_1 and, in addition, two majorants ν_2, ω_2 to replace the prime indicator function. This results in the standard vector-sieve inequality

$$\Lambda_{\mathcal{I}} * \Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}}(X) \geq \Lambda_{\mathcal{I}} * \nu_1 * \omega_2(X) + \Lambda_{\mathcal{I}} * \nu_2 * \omega_1(X) - \Lambda_{\mathcal{I}} * \nu_2 * \omega_2(X).$$

Model functions and the general theorem. Our approach introduces a non-negative model that omits a vector-sieve-type inequality, thereby simplifying the treatment. If we can find two models $T_{\nu}^+ : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ and $T_{\nu} : \mathbb{N} \rightarrow \mathbb{R}$ that are Fourier close to ν , then we could write

$$(1.2) \quad \Lambda_{\mathcal{I}} * \Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}} \geq \Lambda_{\mathcal{I}} * \nu * \Lambda_{\mathcal{K}}$$

$$(1.3) \quad \approx \Lambda_{\mathcal{I}} * T_{\nu}^+ * \Lambda_{\mathcal{K}}$$

$$(1.4) \quad \geq \Lambda_{\mathcal{I}} * T_{\nu}^+ * \omega$$

$$(1.5) \quad \approx \Lambda_{\mathcal{I}} * T_{\nu} * \omega,$$

where (1.2) comes from the non-negativity of prime indicators, and (1.3) comes from the non-negativity of the model. We will explain (1.4) and (1.5) using the appropriate Fourier closeness conditions. The above equations are evaluated at X , and we use $\approx$ to denote equality up to some error specified in Section 2.

The model functions need to be constructed in a way that allows for the calculation of

$$\Lambda_{\mathcal{I}} * T_{\nu} * \omega$$

to be as simple as possible. We need T_{ν}^+ to be non-negative, and we also require some Fourier closeness between ν, T_{ν} and T_{ν}^+ . Later we will see that this condition is closely related to an asymptotic about the expected number of primes, which we will show separately for all three functions. This allows us to formulate the most general version of the theorem, which depends only on the construction of minorants. This way, we separate the circle method proof from sieve theory.

Let, for any $\varepsilon, \varepsilon' > 0$, $Y = X^{\theta_1 + \varepsilon}$ and $H = Y^{\theta_2 + \varepsilon'}$ for some constants $\theta_1, \theta_2 > 0$. First, let us put constraints on the primes accordingly. We will restrict

$$p_1 \in [0.5H, H] := \mathcal{I}, \quad p_2 \in [0.5Y, Y] := \mathcal{J}, \quad p_3 \in [X - Y, X] := \mathcal{K}.$$

We require that the minorant ν is supported in $\mathcal{J}$ and ω is supported in $\mathcal{K}$. If we hope to solve $p_1 + p_2 + p_3 = X$ with these restrictions, we need to ensure that there exists a prime in $\mathcal{K}$. The shortest interval length Y for which we can currently prove the existence of primes in $[X - Y, X]$ is $Y = X^{\frac{21}{40}}$, which gives $\theta_1 = \frac{21}{40}$. Technically, this boils down to constructing a minorant ω , which counts primes well in its interval. Since we will be working with the circle method, it is worth counting primes in arithmetic progressions.

Thus, fix some modulus Q , which will come from the circle method. We will require:

$$(1.6) \quad \left| \sum_{\substack{n \in \mathcal{K} \\ n \equiv l \pmod{q}}} \omega(n) - \frac{c_\omega Y}{\varphi(q)} \right| \ll Y \log^{-B} X, \quad (l, q) = 1, \quad q \leq Q,$$

for some $c_\omega > 0$ and any $B > 0$. If we let p_1 and p_3 vary, then p_2 must live in a short interval of size $\approx H$ in $[0.5Y, Y]$. Hence, we need an asymptotic for primes in short intervals for ν . The right choice is, for a slightly shorter interval length $\tau = \frac{H}{Q} \sim H$

$$(1.7) \quad \int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \nu(n) - \frac{c_\nu \tau}{\varphi(q)} \mathbf{1}_{(l,q)=1} \right|^2 dt \ll \tau^2 Y \log^{-C} X, \quad q \leq Q, \quad l(q),$$

with some $c_\nu > 0$ and any $C > 0$. Currently, the best construction of this minorant comes from a sieving process carried out by Jia [8] with the exponent $\theta_2 = \frac{1}{20}$.

Remark. Runbo Li [9] published a series of three papers titled “Primes in almost all short intervals”, currently under review, in which he pushes the exponent to $\theta_2 = \frac{1}{24}$.

Due to technicalities of the circle method, we also need simple L^2 -bounds and an appropriate support on Q -rough numbers. With this in mind, we can formulate:

Theorem 1.2. *Let $A, \varepsilon', \varepsilon''$ be fixed positive constants and set $Q = \log^{8A} X$. Suppose that there exist constants $\theta_1, \theta_2 > 0$ such that for $Y = X^{\theta_1 + \varepsilon'}$ and $H = Y^{\theta_2 + \varepsilon''}$, one can construct minorants ν and ω bounded by the prime indicators in their respective intervals, which satisfy:*

- (1) **Support:** *Both ν and ω are supported solely on Q -rough numbers, with ν supported in $[0.5Y, Y]$ and ω supported in $[X - Y, X]$;*
- (2) **Asymptotics:** *The minorants satisfy expected short-interval asymptotics for primes (1.6) and (1.7) with positive density constants c_ν, c_ω ;*
- (3) **L^2 bounds:** $\|\nu\|_2, \|\omega\|_2 \ll Y^{\frac{1}{2}} \log^A X$.

Then for any $\varepsilon > 0$ and a sufficiently large odd integer X , the equation $X = p_1 + p_2 + p_3$ with $p_1 \leq H'$ is solvable in primes, where $H' = X^{\theta_1 \theta_2 + \varepsilon}$.

Structure. The framework of the paper is as follows. In the next section, we introduce preliminary results, the appropriate Farey dissection, and then formalise the process described in (1.2) - (1.5). In Section 3, we choose the right model functions and prove the necessary Fourier closeness constraints. Finally, in Section 4, we prove Theorem 1.2, and then import the required minorants from Cai [1] and Grimmelt [5] to prove Theorem 1.1 as a corollary.

2. THE CIRCLE METHOD WITH MINORANTS AND MODELS

Preliminary results. In this short subsection, we recall two classic results which will be useful in our application of the circle method.

Lemma 2.1 (Gallagher’s lemma). *Let $c(n)$ be a sequence with an absolutely convergent Fourier series in $[-T, T]$ for some $T > 0$. Then for $\delta = \frac{1}{T}$,*

$$(2.1) \quad \int_{-T}^T |\widehat{c}(x)|^2 dx \ll \frac{1}{\delta^2} \int_{-\infty}^{\infty} \left| \sum_{y < n \leq y+\delta} c(n) \right|^2 dy.$$

Proof. See [4, Lemma 5.1]. □

Next, we recall a standard inequality.

Lemma 2.2 ($L^2 - L^\infty$ inequality). *Let $E \subset \mathbb{T}$ be a measurable set. Then for any $f \in L^\infty(E)$ and $g, h \in L^2(E)$,*

$$(2.2) \quad \left| \int_E f(x)g(x)h(x) dx \right| \leq \|f\|_\infty \|g\|_2 \|h\|_2.$$

Farey Dissection. Let $X, Y, H > 0$ be given parameters. Set for $A > 0$,

$$Q = \log^{8A} X, \quad \tau = \frac{H}{Q}.$$

For $(a, q) = 1, 1 \leq a \leq q$, let

$$\begin{aligned} \mathfrak{M}(q, a) &= \left(\frac{a}{q} - \frac{1}{\tau}, \frac{a}{q} + \frac{1}{\tau} \right], \\ \mathfrak{M} &= \bigcup_{1 \leq q \leq Q} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(q, a), \\ \mathfrak{m} &= \left(-\frac{1}{\tau}, 1 - \frac{1}{\tau} \right] \setminus \mathfrak{M}, \end{aligned}$$

be the Farey dissection of the unit interval

$$\left(-\frac{1}{\tau}, 1 - \frac{1}{\tau} \right] = \mathfrak{M} \cup \mathfrak{m}.$$

For a convolution $f * g$ of arithmetic functions and any set S , we denote $(f * g)_S$ by

$$(f * g)_S(n) = \int_S \widehat{f}(\alpha) \widehat{g}(\alpha) e(-\alpha n) d\alpha.$$

By Fourier inversion and Farey dissection, we have

$$f * g = (f * g)_\mathfrak{M} + (f * g)_\mathfrak{m}.$$

With this setup, we can now formalise the version of the circle method described in (1.2) - (1.5). The following circle method application is heavily based on [5].

Proposition 2.3 (Circle method). *Let $A > 0$ be a fixed constant. For finitely supported arithmetic functions*

$$(2.3) \quad a, b, c, T_\nu^+ : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$$

$$(2.4) \quad \omega, \nu, T_\nu : \mathbb{N} \rightarrow \mathbb{R},$$

that satisfy the following conditions:

- **Support:** ν, T_ν, T_ν^+ are supported in $[0.5Y, Y]$.
- **Minorisation:** For all n we have

$$(2.5) \quad \nu(n) \leq b(n),$$

$$(2.6) \quad \omega(n) \leq c(n).$$

• **Fourier closeness:** For all $q \leq Q, l(q)$ it holds that

$$(2.7) \quad \int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \nu(n) - T_\nu^+(n) \right|^2 dt \ll \tau^2 Y \log^{-44A} X,$$

$$(2.8) \quad \int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu^+(n) - T_\nu(n) \right|^2 dt \ll \tau^2 Y \log^{-44A} X.$$

• **Minor arcs and trivial bound:** The function a satisfies the following

$$(2.9) \quad \sup_{\alpha \in \mathfrak{m}} |\widehat{a}(\alpha)| \ll H \log^{-3A},$$

$$(2.10) \quad \|\widehat{a}\|_\infty \ll H.$$

• **L^2 bounds:** All functions except a satisfy the L^2 bounds of the form

$$(2.11) \quad \|\cdot\|_2 \ll Y^{\frac{1}{2}} \log^A X.$$

Then we have the following

$$(2.12) \quad a * b * c(X) \geq a * T_\nu * \omega(X) + O(HY \log^{-A} X).$$

Proof. By minorisation and non-negativity of a and c , we have

$$(2.13) \quad a * b * c(X) \geq a * \nu * c(X).$$

Using the $L^2 - L^\infty$ bound, we obtain

$$(a * \nu * c)_\mathfrak{m}(X) = \int_\mathfrak{m} \widehat{a}(\alpha) \widehat{\nu}(\alpha) \widehat{c}(\alpha) e(-X\alpha) d\alpha \ll \sup_{\alpha \in \mathfrak{m}} |\widehat{a}(\alpha)| \cdot \|\nu\|_2 \cdot \|c\|_2 \ll HY \log^{-A} X,$$

where (2.9) and (2.11) have been used. Since $a * \nu * c = (a * \nu * c)_\mathfrak{M} + (a * \nu * c)_\mathfrak{m}$,

$$(2.14) \quad a * \nu * c(X) = (a * \nu * c)_\mathfrak{M}(X) + O(HY \log^{-A} X).$$

Next, we use the Fourier closeness condition. By the trivial bound (2.10), we have

$$\begin{aligned} (2.15) \quad (a * (\nu - T_\nu^+) * c)_{\mathfrak{M}(q,a)}(X) &= \int_{\mathfrak{M}(q,a)} \widehat{a}(\alpha) (\widehat{\nu - T_\nu^+})(\alpha) \widehat{c}(\alpha) e(-X\alpha) d\alpha \\ &\ll H \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} \left| \widehat{\nu} \left(\frac{a}{q} + \lambda \right) - \widehat{T_\nu^+} \left(\frac{a}{q} + \lambda \right) \right| \left| \widehat{c} \left(\frac{a}{q} + \lambda \right) \right| d\lambda \\ &\ll H \left(\int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} \left| \widehat{\nu} \left(\frac{a}{q} + \lambda \right) - \widehat{T_\nu^+} \left(\frac{a}{q} + \lambda \right) \right|^2 d\lambda \right)^{\frac{1}{2}} \left(\int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} \left| \widehat{c} \left(\frac{a}{q} + \lambda \right) \right|^2 d\lambda \right)^{\frac{1}{2}} \\ &\ll H \|c\|_2 \frac{1}{\tau} \left(\int_{\frac{Y}{2}}^Y \left| \sum_{t < n \leq t+\tau} (\nu(n) - T_\nu^+(n)) e_q(an) \right|^2 dt \right)^{\frac{1}{2}} \\ &\ll HY^{\frac{1}{2}} \log^A X \frac{1}{\tau} \left(\int_{\frac{Y}{2}}^Y \left| \sum_{t < n \leq t+\tau} (\nu(n) - T_\nu^+(n)) e_q(an) \right|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and Gallagher's lemma, as well as the support of ν and the model. Lastly, in (2.15) we used the L^2 -bound on c . Now, we sort into residue classes as follows

$$\sum_{t < n \leq t+\tau} (\nu(n) - T_\nu^+(n)) e_q(an) = \sum_{l(q)} e_q(al) \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} (\nu(n) - T_\nu^+(n)).$$

Applying the Cauchy-Schwarz inequality, we get

$$\left| \sum_{l(q)} e_q(al) \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} (\nu(n) - T_\nu^+(n)) \right|^2 \leq q \sum_{l(q)} \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} (\nu(n) - T_\nu^+(n)) \right|^2.$$

Inserting this bound back into (2.15), we can now directly apply the Fourier closeness condition (2.7)

$$(a * (\nu - T_\nu^+) * c)_{\mathfrak{M}(q,a)}(X) \ll HY^{\frac{1}{2}} \log^A X \frac{1}{\tau} \left(q\tau^2 Y \log^{-44A} X \right)^{\frac{1}{2}} \ll HY q^{\frac{1}{2}} \log^{-21A} X.$$

Finally, we obtain

$$\begin{aligned} (a * (\nu - T_\nu^+) * c)_{\mathfrak{M}}(X) &= \sum_{q \leq Q} \sum_{a(q)^*} (a * (\nu - T_\nu^+) * c)_{\mathfrak{M}(q,a)}(X) \\ &\ll HY \log^{-21A} X \sum_{q \leq Q} \varphi(q) q^{\frac{1}{2}} \ll Q^{\frac{5}{2}} HY \log^{-21A} X = HY \log^{-A} X. \end{aligned}$$

Therefore,

$$(2.16) \quad (a * \nu * c)_{\mathfrak{M}}(X) = (a * T_\nu^+ * c)_{\mathfrak{M}}(X) + O(HY \log^{-A} X).$$

Lastly, by the same arguments that led up to 2.14, we have

$$(2.17) \quad a * T_\nu^+ * c(X) = (a * T_\nu^+ * c)_{\mathfrak{M}}(X) + O(HY \log^{-A} X).$$

Combining (2.13) -(2.17), we get

$$a * b * c(X) \geq a * T_\nu^+ * c(X) + O(HY \log^{-A} X).$$

Now, since T_ν^+ is non-negative, we can identically prove the same estimates as above, replacing c by its minorant ω and then replacing the non-negative model with T_ν , to obtain

$$a * T_\nu^+ * c(X) \geq a * T_\nu * \omega(X) + O(HY \log^{-A} X).$$

Putting the above inequalities together, we finally get

$$a * b * c(X) \geq a * T_\nu * \omega(X) + O(HY \log^{-A} X).$$

□

3. CHOICE OF MODEL FUNCTIONS

We now turn to choosing the right model functions that will be used to prove the theorem using the circle method machinery described in Section 2. Suppose that we have chosen the minorant ν as in Proposition 2.3. We import the choice of T_ν and T_ν^+ from [5]. It is motivated by two things. First, we want the calculation of the triple convolution

$$\Lambda_{\mathcal{I}} * T_\nu * \omega$$

to be as simple as possible and mostly supported on major arcs. Second, we want this function to carry information about the primes. The standard choice is thus

$$\Lambda_Q(n) := \sum_{k \leq Q} \sum_{a(k)^*} \frac{\mu(k)}{\varphi(k)} e_k(an).$$

We scale this function to construct the model. For n in $[0.5Y, Y]$ we pick

$$T_\nu(n) = c_\nu \Lambda_Q(n),$$

where $c_\nu > 0$ is some constant chosen later that depends on the density of ν . We further set $T_\nu(n) = 0$ for other n . To prove the Fourier closeness condition, we will utilise the lemma below.

Lemma 3.1. *Let T_ν be as above; for any $B > 0$ and for all $q \leq Q$, $l(q)$,*

$$(3.1) \quad \int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu(n) - \frac{c_\nu \tau}{\varphi(q)} \mathbf{1}_{(l,q)=1} \right|^2 dt \ll \tau^2 Y \log^{-B} X.$$

Proof. We need to analyse the inner sum for a fixed t :

$$\begin{aligned} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu(n) &= c_\nu \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \sum_{k \leq Q} \frac{\mu(k)}{\varphi(k)} \sum_{a(k)^*} e_k(an) \\ &= c_\nu \sum_{k \leq Q} \frac{\mu(k)}{\varphi(k)} \sum_{a(k)^*} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} e_k(an) \\ &= c_\nu \sum_{k \leq Q} \frac{\mu(k)}{\varphi(k)} \sum_{a(k)^*} e_k(al) \sum_{\substack{t-l < m \leq t+\tau-l \\ q \mid m}} e_k(amq). \end{aligned}$$

Now divide this into cases $k \mid q$ and $k \nmid q$, which will yield the main term and the error term. For the error term where $k \nmid q$, we use the standard exponential sum bound to obtain

$$\begin{aligned} c_\nu \sum_{\substack{k \leq Q \\ k \nmid q}} \frac{\mu(k)}{\varphi(k)} \sum_{a(k)^*} e_k(al) \sum_{\substack{t-l < m \leq t+\tau-l \\ q \mid m}} e_k(amq) &\ll c_\nu \sum_{\substack{k \leq Q \\ k \nmid q}} \frac{|\mu(k)|}{\varphi(k)} \sum_{a(k)^*} \frac{1}{\|aq/k\|} \\ &\leq c_\nu \sum_{k \nmid q, k \leq Q} \frac{|\mu(k)|}{\varphi(k)} \sum_{a(k)^*} \frac{k}{(k, q)} \\ (3.2) \quad &= c_\nu \sum_{k \nmid q, k \leq Q} |\mu(k)| \frac{k}{(k, q)} \ll Q^2, \end{aligned}$$

where in (3.2) we have used the simple estimate $\frac{1}{\|aq/k\|} \leq \frac{k}{(k,q)}$. For the main term where $k \mid q$, we have $e_k(amq) = 1$ for all m . Thus, the sum over m counts integers in the interval of length $\frac{\tau}{q}$, giving $\frac{\tau}{q} + O(1)$. The main term is therefore

$$c_\nu \sum_{k|q} \frac{\mu(k)}{\varphi(k)} \left(\sum_{a(k)^*} e_k(al) \right) \left(\frac{\tau}{q} + O(1) \right) = c_\nu \left(\frac{\tau}{q} + O(1) \right) \sum_{k|q} \frac{\mu(k)}{\varphi(k)} c_k(l),$$

where $c_k(l)$ is the Ramanujan sum. Since $\frac{\mu(k)}{\varphi(k)} c_k(l)$ is a multiplicative function of k , so is the sum over the divisors $\sum_{k|q} \frac{\mu(k)}{\varphi(k)} c_k(l)$. Since if k is not square-free $\mu(k) = 0$, the sum is supported on square-free divisors. We can then express the sum as a product

$$\sum_{k|q} \frac{\mu(k)}{\varphi(k)} c_k(l) = \prod_{p|q} \left(1 + \frac{\mu(p)}{\varphi(p)} c_p(l) \right).$$

We now consider two cases based on the coprimality of l and q :

Case 1: $(l, q) = 1$. For every prime $p \mid q$, we have $p \nmid l$. Therefore, $c_p(l) = \mu(p) = -1$. The Euler product evaluates to

$$\prod_{p|q} \left(1 + \frac{(-1)}{p-1} (-1) \right) = \prod_{p|q} \frac{p}{p-1} = \frac{q}{\varphi(q)}.$$

Substituting this in, the main term becomes $c_\nu \frac{q}{\varphi(q)} \left(\frac{\tau}{q} + O(1) \right) = \frac{c_\nu \tau}{\varphi(q)} + O(Q)$, which matches the expected main term in (3.1).

Case 2: $(l, q) > 1$. Since $q \leq Q$, there is at least one prime $p_0 \leq Q$ such that $p_0 \mid q$ and $p_0 \mid l$. For this prime, the Ramanujan sum evaluates to $c_{p_0}(l) = \varphi(p_0) = p_0 - 1$. The corresponding factor in the Euler product is

$$1 + \frac{\mu(p_0)}{\varphi(p_0)} c_{p_0}(l) = 1 + \frac{-1}{p_0 - 1} (p_0 - 1) = 0.$$

Thus, the entire product vanishes. The main term is exactly 0, matching the indicator $\mathbf{1}_{(l,q)=1}$.

In both cases, subtracting the expected main term leaves an error of size $O(Q^2)$. Integrating the square of this error gives

$$\int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu(n) - \frac{c_\nu \tau}{\varphi(q)} \mathbf{1}_{(l,q)=1} \right|^2 dt \ll YQ^4 \ll \tau^2 Y \log^{-B} X,$$

where the last inequality follows from the fact that $\tau = \frac{H}{Q}$ and H is polynomial in X , whereas Q is a power of $\log X$. This completes the proof. $\square$

Next, we construct the non-negative model. One way is to choose a scaled Q -rough number indicator. However, we choose the approach that utilises an upper-bound sieve. We work with the β -sieve as described in [6, Chapter 6.4]. Let $\Omega(n) = \Omega_{D,z}(n)$ be an upper bound β -sieve for $\beta = 10$ with sifting range z and level D . This means that Ω is non-negative. We write

$$\Omega_{D,z}(n) = \sum_{d|n} \lambda_d,$$

where λ_d are the associated sieve weights that satisfy $|\lambda_d| \leq 1$ and are supported only on $d \leq D$. We now choose T_ν^+ . Set $z = Q$, $D = H^{\frac{1}{2}}$ and for n in $[0.5Y, Y]$ put

$$T_\nu^+(n) = c_\nu \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \Omega_{D,z}(n)$$

where c_ν is a constant as before. Further, we set $T_\nu^+(n) = 0$ for other n . Since $\Omega(n) \geq 0$, $T_\nu^+(n)$ is non-negative. We now prove the same result as Lemma 3.1 but for this model.

Lemma 3.2. *Let T_ν^+ be as above; for any $B > 0$ and for all $q \leq Q$, $l(q)$,*

$$(3.3) \quad \int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu^+(n) - \frac{c_\nu \tau}{\varphi(q)} \mathbf{1}_{(l,q)=1} \right|^2 dt \ll \tau^2 Y \log^{-B} X.$$

Proof. As before, we need to analyse the inner sum for a fixed t :

$$\sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu^+(n) = c_\nu \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \Omega_{D,z}(n).$$

Now we split this into cases based on whether $(l, q) = 1$. We use the two results obtained by Grimmelt in the proof of [5, Lemma 4.1].

Case 1: $(l, q) = 1$. In this case, we get the following main term and error:

$$\begin{aligned} c_\nu \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \Omega_{D,z}(n) &= \frac{c_\nu \tau}{\varphi(q)} \left(1 + O(e^{-\frac{\log D}{\log Q}})\right) + O(D) \\ &\ll \frac{c_\nu \tau}{\varphi(q)} + \tau e^{-\frac{\log D}{\log Q}} + qD. \end{aligned}$$

Case 2: $(l, q) > 1$. We get the following error:

$$c_\nu \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \sum_{\substack{l(q) \\ (l,q) \neq 1}} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \Omega_{D,z}(n) = O(\tau e^{-\frac{\log D}{\log Q}} + qD)$$

And hence, since we are interested in just one non-coprime class and everything in the sum is non-negative, we get for fixed $(l, q) > 1$,

$$c_\nu \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-1} \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} \Omega_{D,z}(n) \ll \tau e^{-\frac{\log D}{\log Q}} + qD.$$

In both cases, when we subtract the main term given by the indicator function, we obtain

$$\int_{\frac{Y}{2}}^Y \left| \sum_{\substack{t < n \leq t+\tau \\ n \equiv l \pmod{q}}} T_\nu^+(n) - \frac{c_\nu \tau}{\varphi(q)} \mathbf{1}_{(l,q)=1} \right|^2 dt \ll Y \left(\tau e^{-\frac{\log D}{\log Q}} + qD \right)^2 \ll \tau^2 Y \log^{-B} X.$$

Since $D = H^{\frac{1}{2}}$, the exponential term decays faster than any logarithmic term and $(qD)^2 \ll Q^2 H \ll \tau^2 \log^{-B} X$ for any $B > 0$. This completes the proof. $\square$

4. PROOF OF THEOREM 1.2

To prove the theorem, we use the modified circle method and the previously explained models. At the end, we import the minorants from [1] and [5] to prove Theorem 1.1. We require a small lemma about the minor arc bound of $\Lambda_{\mathcal{I}}$.

Lemma 4.1. *The log-weighted prime indicator $\Lambda_{\mathcal{I}}$ satisfies the following bounds:*

- (i) $\widehat{\Lambda_{\mathcal{I}}}(\alpha) \ll H \log^{-3A} X$, $\alpha \in \mathfrak{m}$,
- (ii) $\|\widehat{\Lambda_{\mathcal{I}}}\|_{\infty} \ll H$.

Proof. For (i) see [2]. Property (ii) is a simple consequence of the prime number theorem. $\square$

Now we have all the necessary tools for the final proof.

Proof of Theorem 1.2. Let ν, ω be the described minorants. For simplicity, put $B = 25A$ and $C = 44A$ in (1.6) and (1.7) respectively. First, construct T_{ν} , T_{ν}^+ accordingly, as explained in Section 3. We apply Proposition 2.3 with

$$a = \Lambda_{\mathcal{I}}, \quad b = \Lambda_{\mathcal{J}}, \quad c = \Lambda_{\mathcal{K}}.$$

The only conditions to check which do not immediately follow from Lemma 3.1, Lemma 3.2, Lemma 4.1 and the assumptions in the theorem are the Fourier closeness conditions (2.7) and (2.8). However, they simply follow by applying the trivial inequality $2|x|^2 + 2|y|^2 \geq |x+y|^2$ to (1.7), (3.1) and (3.3). We also need to check the support of the models, as well as the L^2 bounds for the models and the prime indicators, but they follow from their construction. Thus, we obtain

$$\Lambda_{\mathcal{I}} * \Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}}(X) \geq \Lambda_{\mathcal{I}} * T_{\nu} * \omega(X) + O(HY \log^{-A} X).$$

We are left with finding a lower bound for $\Lambda_{\mathcal{I}} * T_{\nu} * \omega(n)$ for odd X . We have

$$\begin{aligned} \Lambda_{\mathcal{I}} * T_{\nu} * \omega(X) &= \sum_{\substack{p_1+n_2+n_3=X \\ p_1 \in \mathcal{I}, n_2 \in \mathcal{J}, n_3 \in \mathcal{K}}} \log p_1 T_{\nu}(n_2) \omega(n_3) = c_{\nu} \sum_{p_1 \in \mathcal{I}, n \in \mathcal{K}} \log p_1 \omega(n) \Lambda_Q(X - p_1 - n) \\ &= c_{\nu} \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{a(q)^*} e_q(aX) \left(\sum_{p_1 \in \mathcal{I}} \log p_1 e_q(-ap_1) \right) \left(\sum_{n \in \mathcal{K}} \omega(n) e_q(-an) \right). \end{aligned}$$

First, we estimate for $q \leq Q$, $(a, q) = 1$ using the Siegel-Walfisz theorem:

$$\begin{aligned} \sum_{p_1 \in \mathcal{I}} \log p_1 e_q(-ap_1) &= \sum_{l(q)^*} e_q(-al) \sum_{\substack{p_1 \in \mathcal{I} \\ p_1 \equiv l \pmod{q}}} \log p_1 \\ &= \sum_{l(q)^*} e_q(-al) \left(\frac{H}{2\varphi(q)} + O(H \log^{-9A} X) \right) \\ &= \frac{\mu(q)H}{2\varphi(q)} + O(\varphi(q)H \log^{-9A} X), \end{aligned}$$

where we used the fact that $Q \ll H$ so that $(p_1, q) = 1$. Similarly, because ω is supported on Q -rough numbers, using (1.6) we have

$$\begin{aligned} \sum_{n \in \mathcal{K}} \omega(n) e_q(-an) &= \sum_{l(q)^*} e_q(-al) \sum_{\substack{n \in \mathcal{K} \\ n \equiv l \pmod{q}}} \omega(n) \\ &= \sum_{l(q)^*} e_q(-al) \left(\frac{c_\omega Y}{\varphi(q)} + O(Y \log^{-25A} X) \right) = \frac{c_\omega \mu(q) Y}{\varphi(q)} + O(\varphi(q) Y \log^{-9A} X). \end{aligned}$$

Combining all of this and dropping lower-order error terms, we get

$$\begin{aligned} \Lambda_{\mathcal{I}} * T_\nu * \omega(X) &= c_\nu \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \sum_{a(q)^*} e_q(aX) \left(\frac{c_\omega \mu(q)^2 H Y}{2 \varphi(q)^2} + O(H Y \log^{-9A} X) \right) \\ &= \frac{1}{2} c_\nu c_\omega H Y (\mathfrak{S}(X) + O(Q^{-1})) + O(Q H Y \log^{-9A} X) \\ &= \frac{1}{2} c_\nu c_\omega H Y \mathfrak{S}(X) + O(H Y \log^{-A} X), \end{aligned}$$

where

$$\mathfrak{S}(X) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)^3} \sum_{a(q)^*} e_q(aX).$$

By the well-known fact that for odd X we have $\mathfrak{S}(X) > \frac{1}{2}$, we conclude that for large enough odd $X > 0$,

$$\begin{aligned} \Lambda_{\mathcal{I}} * \Lambda_{\mathcal{J}} * \Lambda_{\mathcal{K}}(X) &\geq \Lambda_{\mathcal{I}} * T_\nu * \omega(X) + O(H Y \log^{-A} X) \\ &= \frac{1}{2} c_\nu c_\omega H Y \mathfrak{S}(X) + O(H Y \log^{-A} X) > 0, \end{aligned}$$

which finishes the proof. $\square$

Lastly, we prove Theorem 1.1 as a corollary of Theorem 1.2.

Proof of Theorem 1.1. We need to show that the assumptions in Theorem 1.2 hold for $\theta_1 = \frac{21}{40}$ and $\theta_2 = \frac{1}{20}$. The necessary work has already been done by Grimmelt [5] and Cai [1]. For ν , we use the same minorant as Cai, scaled by a factor of $\log 0.5Y$, to ensure that it is a minorant for the weighted prime indicator. This drops a factor of $\log Y$ in the main term of (1.7). (Compare with [1, Lemma 1 (iv)]). For $(l, q) = 1$, the condition is satisfied exactly as shown by Cai. Else, since ν is supported only on Q -rough numbers, the sum vanishes completely, and so does the main term, making the condition trivially satisfied. Since we are saving any logarithmic power of X , we can choose the exponent to be any $C > 0$ instead of $-40A$ as Cai did. The remaining properties are naturally satisfied.

For the second minorant ω , we use the same construction as Harman in [6, Sections 7.4 to 7.9]. For our interval, we use $\mathcal{K} = [X - Y, X]$ and set ω to be $\log(X - Y)$ times Harman's function. As proved by Grimmelt [5, Lemma 5.1], this is a minorant for the weighted prime indicator; it has the right support, and its coefficient bounds yield exactly the needed L^2 bound. Lastly, it gives a short character condition which states that there is some $c_\omega > 0.09$ such that for any $B > 0$ and any character $\chi(q)$ with $q \leq Q$ we have

$$(4.1) \quad E(\chi) = \sum_{n \in \mathcal{K}} (\omega(n) \chi(n) - c_\omega \delta_\chi) \ll Y \log^{-B-8A} X.$$

This is exactly the condition we need for the asymptotics of the expected number of primes. To see this, we use the orthogonality of characters, ie. for $(l, q) = 1$,

$$\mathbf{1}_{n \equiv l \pmod{q}} = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) \chi(n).$$

And thus

$$\begin{aligned} \sum_{\substack{n \in \mathcal{K} \\ n \equiv l \pmod{q}}} \omega(n) &= \sum_{n \in \mathcal{K}} \omega(n) \mathbf{1}_{n \equiv l \pmod{q}} = \frac{1}{\varphi(q)} \sum_{n \in \mathcal{K}} \omega(n) \sum_{\chi \pmod{q}} \bar{\chi}(l) \chi(n) \\ &= \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) \sum_{n \in \mathcal{K}} \omega(n) \chi(n) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) (c_\omega \delta_\chi Y + E(\chi)) \\ &= \frac{c_\omega Y}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) E(\chi). \end{aligned}$$

Since the characters are roots of unity, combining the above with (4.1), we obtain

$$\left| \sum_{\substack{n \in \mathcal{K} \\ n \equiv l \pmod{q}}} \omega(n) - \frac{c_\omega Y}{\varphi(q)} \right| \leq \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} |E(\chi)| \ll Y \log^{-B} X,$$

which finishes the minorant construction. The proof now follows immediately by Theorem 1.2. $\square$

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